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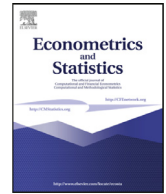
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Hypothesis testing for tail dependence parameters on the boundary of the parameter space

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ABSTRACT

Modelling multivariate tail dependence is one of the key challenges in extreme-value theory. Multivariate extremes are usually characterized using parametric models, some of which have simpler submodels at the boundary of their parameter space. Hypothesis tests are proposed for tail dependence parameters that, under the null hypothesis, are on the boundary of the alternative hypothesis. The asymptotic distribution of the weighted least squares estimator is given when the true parameter vector is on the boundary of the parameter space, and two test statistics are proposed. The performance of these test statistics is evaluated for the Brown–Resnick model and the max-linear model. In particular, simulations show that it is possible to recover the optimal number of factors for a max-linear model. Finally, the methods are applied to characterize the dependence structure of two major stock market indices, the DAX and the CAC40.

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1. Introduction

Extreme-value theory is the branch of statistics concerned with the characterization of extreme events. These occur in a large variety of fields, such as hydrology, meteorology, finance and insurance, but also in settings like human life span or athletic records. For some examples, see Einmahl and Magnus (2008), Towler et al. (2010), Chavez-Demoulin et al. (2016), Thomas et al. (2016) and Rootzén and Zholud (2017). In the univariate case, the limiting distribution of suitably normalized block maxima or threshold exceedances can be characterized entirely using the generalized extreme-value (Fisher and Tippett, 1928; Gnedenko, 1943) and the generalized Pareto distribution (Balkema and De Haan, 1974; Pickands et al., 1975) respectively. However, many extreme events are inherently multivariate, and an important challenge is to model the tail dependence between two or more random variables of interest. If dependence disappears as the variables take on more and more extreme values, we say that they are asymptotically independent. Testing for asymptotic independence and modelling asymptotically independent data has been done in Ledford and Tawn (1996), Draisma et al. (2004), Hüsler and Li (2009), Wadsworth et al. (2016) and Guillou et al. (2018), among others. In the following, we consider the framework of asymptotic dependence.

The family of limiting distributions for multivariate maxima or threshold exceedances is infinite-dimensional, so that realistic and computationally feasible parametric models need to be proposed. Examples include Gumbel (1960), Smith (1990), Tawn (1990) and Kabluchko et al. (2009). Many different approaches of estimating tail dependence parameters exist. Fitting a multivariate extreme-value distribution to componentwise block maxima is done in Padoan et al. (2010), Davison et al. (2012), Castruccio et al. (2016) and Dombry et al. (2017), along others. Multivariate threshold exceedances, which are the

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focus of this work, can be tackled using the stable tail dependence function (Drees and Huang, 1998) or the multivariate generalized Pareto distribution (Rootzén and Tajvidi, 2006). Common estimation methods include maximum likelihood (Wadsworth and Tawn, 2014; Kiriliouk et al., 2018a; de Fondeville and Davison, 2018) and minimum distance estimation (Einmahl et al., 2012; 2016; 2018).

Some popular parametric models have simpler submodels at the boundary of the parameter space. The main goal of this paper is to propose hypothesis tests for tail dependence parameters that, under the null hypothesis, are on the boundary of the alternative hypothesis. A first example is the Brown–Resnick model (Kablichko et al., 2009), popular in spatial extremes, which reduces to the simpler Smith model (Smith, 1990) when the shape parameter of the former attains its upper bound. A second example is the max-linear model, where each component of a vector can be interpreted as the maximum shock among a set of independent factors. Recent work on max-linear models is numerous. In Gissibl and Klüppelberg (2018) and Gissibl et al. (2018), a recursive max-linear model on a directed acyclic graph is considered. In Cui and Zhang (2018) and Zhao and Zhang (2018), max-linear models are used to model a latent factor structure for financial returns. Finally, the Marshall–Olkin model (Embrechts et al., 2003; Segers, 2012) is a submodel that is frequently used in practice, see for instance Burtschell et al., (2009), Su and Furman, (2017) or Brigo et al., (2018).

Because of their non-differentiability, max-linear models cannot be estimated by standard likelihood methods and hence we use the weighted least squares estimator of Einmahl et al. (2018), which is centred around the stable tail dependence function. The hypothesis testing framework described in that paper is valid for parameters in the interior of the parameter space, and hence it cannot be used to test whether one or more factor loadings are equal to zero. The hypothesis tests proposed in this paper can be applied to decide how many factors are necessary to model a certain dataset or to test if a more specific model (eg, Marshall–Olkin) is sufficient, bypassing the limitation that the number of factors has to be chosen upfront.

The paper is organized as follows. Section 2 presents background on multivariate extremes and introduces the weighted least squares estimator. Section 3 establishes asymptotic normality for the weighted least squares estimator when the true parameter vector lies on the boundary of the parameter space. Using results from Andrews (1999), Andrews (2001) and Andrews (2002), a deviance- and a Wald-type test statistic are proposed, whose asymptotic distributions are easily computable when the dimension of the parameter vector on the boundary is moderate. In Section 4.2, we perform simulations to evaluate the performance of the test statistics for the Brown–Resnick and the max-linear models. Finally, Section 5 illustrates our methods on two major stock market indices, the DAX and the CAC40. Technical definitions and proofs are deferred to the appendix.

2. Background

2.1. Multivariate extreme-value theory

Let $\mathbf{X}_i = (X_{i1}, \dots, X_{id})$, $i \in \{1, \dots, n\}$, be random vectors in \mathbb{R}^d with cumulative distribution function F and marginal cumulative distribution functions F_1, \dots, F_d . Let $\mathbf{M}_n := (M_{n1}, \dots, M_{nd})$ with $M_{nj} := \max(X_{1j}, \dots, X_{nj})$ for $j = 1, \dots, d$. We say that F is in the max-domain of attraction of an extreme-value distribution G if there exist sequences of normalizing constants $\mathbf{a}_n = (a_{n1}, \dots, a_{nd}) > 0$ and $\mathbf{b}_n = (b_{n1}, \dots, b_{nd}) \in \mathbb{R}^d$ such that

$$\mathbb{P}\left[\frac{\mathbf{M}_n - \mathbf{b}_n}{\mathbf{a}_n} \leq \mathbf{x}\right] = F^n(\mathbf{a}_n \mathbf{x} + \mathbf{b}_n) \xrightarrow{d} G(\mathbf{x}), \quad \text{as } n \rightarrow \infty. \quad (1)$$

The margins, G_1, \dots, G_d , of G are univariate extreme-value distributions,

$$G_j(x_j) = \exp \left\{ - \left(1 + \gamma_j \frac{x_j - \mu_j}{\sigma_j} \right)_+^{-1/\gamma_j} \right\}, \quad \sigma_j > 0, \gamma_j, \mu_j \in \mathbb{R},$$

where $x_+ := \max(x, 0)$. The function G is determined by

$$G(\mathbf{x}) = \exp \{ -\ell(-\log G_1(x_1), \dots, -\log G_d(x_d)) \},$$

where $\ell: [0, \infty)^d \rightarrow [0, \infty)$ is called the (stable) tail dependence function,

$$\ell(\mathbf{x}) := \lim_{t \downarrow 0} t^{-1} \mathbb{P}[1 - F_1(X_{11}) \leq tx_1 \text{ or } \dots \text{ or } 1 - F_d(X_{1d}) \leq tx_d]. \quad (2)$$

The cumulative distribution function F is in the max-domain of attraction of a d -variate extreme value distribution G if and only if the limit in (2) exists and the marginal distributions in (1) converge to univariate extreme-value distributions. In what follows, we only assume existence of (2), which concerns the dependence structure of F , but not the marginal distributions F_1, \dots, F_d .

Because the class of stable tail dependence functions is infinite-dimensional, one usually considers parametric models for ℓ . Henceforth we assume that ℓ belongs to a parametric family $\{\ell(\cdot; \boldsymbol{\theta}); \boldsymbol{\theta} \in \Theta\}$ with $\Theta \subset \mathbb{R}^p$. Some examples can be found below, see also de Haan and Ferreira (2006), Falk et al. (2010), Segers (2012) and references therein.

Example 2.1. The d -dimensional logistic model (Gumbel, 1960) has stable tail dependence function

$$\ell(\mathbf{x}; \boldsymbol{\theta}) = (x_1^{1/\theta} + \dots + x_d^{1/\theta})^\theta, \quad \boldsymbol{\theta} \in (0, 1].$$

If $\theta = 1$, the variables are (asymptotically) independent, while $\theta \downarrow 0$ corresponds to the situation of complete dependence.

Example 2.2. The d -dimensional *Brown–Resnick model* defined on spatial locations $\mathbf{s}_1, \dots, \mathbf{s}_d \in \mathbb{R}^2$ has stable tail dependence function

$$\ell(\mathbf{x}; \boldsymbol{\theta}) = \sum_{j=1}^d x_j \Phi_{d-1}(\eta^{(j)}(1/\mathbf{x}); \Upsilon^{(j)}),$$

where $\eta^{(j)}(\mathbf{x}) = (\eta_1^{(j)}(x_1, x_j), \dots, \eta_{j-1}^{(j)}(x_{j-1}, x_j), \eta_{j+1}^{(j)}(x_{j+1}, x_j), \dots, \eta_d^{(j)}(x_d, x_j)) \in \mathbb{R}^{d-1}$,

$$\eta_l^{(j)}(x_l, x_j) = \sqrt{\frac{\gamma(\mathbf{s}_j - \mathbf{s}_l)}{2}} + \frac{\log(x_l/x_j)}{\sqrt{2\gamma(\mathbf{s}_j - \mathbf{s}_l)}} \in \mathbb{R},$$

$\gamma(\mathbf{s}) = (||\mathbf{s}||/\rho)^\alpha$ and $\Upsilon^{(l)} \in \mathbb{R}^{(d-1) \times (d-1)}$ is the correlation matrix with entries

$$\Upsilon_{lk}^{(j)} = \frac{\gamma(\mathbf{s}_j - \mathbf{s}_l) + \gamma(\mathbf{s}_j - \mathbf{s}_k) - \gamma(\mathbf{s}_l - \mathbf{s}_k)}{2\sqrt{\gamma(\mathbf{s}_j - \mathbf{s}_l)\gamma(\mathbf{s}_j - \mathbf{s}_k)}}, \quad l, k = 1, \dots, d; l, k \neq j,$$

(Kabluchko et al., 2009; Huser and Davison, 2013). The parameter vector is $\boldsymbol{\theta} = (\rho, \alpha) \in (0, \infty) \times (0, 2]$. The *Smith model* (Smith, 1989) is obtained when $\alpha = 2$.

Example 2.3. The *max-linear model* with r factors has stable tail dependence function

$$\ell(\mathbf{x}; \boldsymbol{\theta}) = \sum_{t=1}^r \max_{j=1, \dots, d} b_{jt} x_j, \quad \mathbf{x} \in [0, \infty)^d, \quad (3)$$

where the factor loadings b_{jt} are non-negative constants such that $\sum_{t=1}^r b_{jt} = 1$ for every $j \in \{1, \dots, d\}$ and all column sums of the $d \times r$ matrix $B := (b_{jt})_{j,t}$ are positive (Einmahl et al., 2012). Since the rows of B sum up to one, the parameter matrix has only $d \times (r-1)$ free elements. Rearranging the columns of B will not change the value of the stable tail dependence function. For identification purposes, we define the parameter vector $\boldsymbol{\theta}$ by stacking the columns of B in decreasing order of their sums, leaving out the column with the lowest sum. An example of a random vector $\mathbf{Z} = (Z_1, \dots, Z_d)$ that has stable tail dependence function (3) is

$$Z_j = \max_{t=1, \dots, r} b_{jt} S_t, \quad \text{for } j \in \{1, \dots, d\},$$

where S_1, \dots, S_r are independent unit Fréchet variables, $\mathbb{P}[S_j \leq x] = \exp(-1/x)$ for $x > 0$ and $j \in \{1, \dots, d\}$.

Example 2.4. Let I_1, \dots, I_d denote random (possibly dependent) Bernoulli random variables with $p_j = \mathbb{P}[I_j = 1] \in (0, 1]$ for $j \in \{1, \dots, d\}$. Let, for $J \subset \{1, \dots, d\}$, $p(J) = \mathbb{P}[\{j = 1, \dots, d : I_j = 1\} = J]$, so that $(p(J))_{J \subset \{1, \dots, d\}}$ is a probability distribution. The *multivariate Marshall–Olkin model* (Embrechts et al., 2003; Segers, 2012) has stable tail dependence function

$$\ell(\mathbf{x}; \boldsymbol{\theta}) = \sum_{\emptyset \neq J \subset \{1, \dots, d\}} p(J) \max_{j \in J} \left(\frac{x_j}{p_j} \right).$$

In this model, any subset of components of the vector is assigned a shock that influences all components of that subset. An interpretation in terms of credit risk modelling can be found in Embrechts et al. (2003).

The Marshall–Olkin model is obtained as a special case of the max-linear model (3) by setting $b_{jt} = \{p(J_t)/p_j\} \mathbb{1}(j \in J_t)$, where $J_t \in \mathcal{P}(\{1, \dots, d\})$, $J_t \neq \emptyset$, and $\mathcal{P}(A)$ denotes the power set of A . The Marshall–Olkin model is well-known in $d = 2$, where

$$B = \begin{pmatrix} b_{11} & 0 & 1 - b_{11} \\ 0 & b_{22} & 1 - b_{22} \end{pmatrix} = \begin{pmatrix} \frac{P[I_1 = 1, I_2 = 0]}{P[I_1 = 1]} & 0 & \frac{P[I_1 = 1, I_2 = 1]}{P[I_1 = 1]} \\ 0 & \frac{P[I_1 = 0, I_2 = 1]}{P[I_2 = 1]} & \frac{P[I_1 = 1, I_2 = 1]}{P[I_2 = 1]} \end{pmatrix}.$$

2.2. Estimation of the stable tail dependence function

2.2.1. Nonparametric estimation of the stable tail dependence function

Let $k = k_n \in (0, n]$ be such that $k \rightarrow \infty$ and $k/n \rightarrow 0$ as $n \rightarrow \infty$. A straightforward nonparametric estimator of ℓ is obtained by replacing \mathbb{P} and F_1, \dots, F_d in (2) by the (modified) empirical distribution functions and replacing t by k/n , yielding

$$\tilde{\ell}_{n,k}(x) := \frac{1}{k} \sum_{i=1}^n \mathbb{1}\{R_{i1} > n + 1/2 - kx_1 \text{ or } \dots \text{ or } R_{id} > n + 1/2 - kx_d\}.$$

Here, $R_{ij} = \sum_{t=1}^n \mathbb{1}\{X_{tj} \leq X_{ij}\}$ denotes the rank of X_{ij} among X_{1j}, \dots, X_{nj} . This estimator, the *empirical tail dependence function*, was introduced in slightly different form for $d = 2$ in Huang (1992) and studied further in Drees and Huang (1998). Using

$n + 1/2$ rather than n allows for better finite-sample properties, that is, for a lower mean squared error (Einmahl et al., 2012).

Another nonparametric estimator is the *beta tail dependence function*, which was very recently proposed in Kiriliouk et al. (2018b). Contrary to the empirical tail dependence function, the beta tail dependence function is a smooth estimator which leads to some improvement in its finite-sample behavior. Its name stems from the fact that it is based on the empirical beta copula (Segers et al., 2017). We define

$$\tilde{\ell}_{n,k}^{\beta}(\mathbf{x}) = \frac{n}{k} \left\{ 1 - \mathbb{C}_n^{\beta} \left(1 - \frac{k\mathbf{x}}{n} \right) \right\}, \quad \text{where} \quad \mathbb{C}_n^{\beta}(\mathbf{u}) = \frac{1}{n} \sum_{i=1}^n \prod_{j=1}^d F_{n,R_{ij}}(u_j),$$

and for $r \in \{1, \dots, n\}$, $F_{n,r}(u) = \sum_{s=1}^r \binom{n}{s} u^s (1-u)^{n-s}$ is the cumulative distribution function of a Beta($r, n+1-r$) random variable.

A drawback of $\tilde{\ell}_{n,k}$ or $\tilde{\ell}_{n,k}^{\beta}$ might be their possibly growing bias as k increases: Fougères et al. (2015) show that, under suitable conditions, the estimator $\tilde{\ell}_{n,k}$ satisfies the asymptotic expansion

$$\tilde{\ell}_{n,k}(\mathbf{x}) - \ell(\mathbf{x}) \approx k^{-1/2} Z_{\ell}(\mathbf{x}) + \alpha(n/k)M(\mathbf{x}),$$

where Z_{ℓ} is a continuous centered Gaussian process, α is a function such that $\lim_{x \rightarrow \infty} \alpha(x) = 0$ and M is a continuous function. When $\sqrt{k}\alpha(n/k)$ tends to a non-zero constant, an asymptotic bias appears. In Fougères et al. (2015) and Beirlant et al. (2016), this bias is estimated and subtracted from the estimator $\tilde{\ell}_{n,k}(\mathbf{x})$ in order to propose new bias-corrected estimators.

Finally, many other nonparametric estimators exist of the Pickands dependence function, which is the restriction of ℓ to the unit simplex. See, for instance, Capérea et al. (1997), Zhang et al. (2008), Gudendorf and Segers (2012), Berghaus et al. (2013), Vettori et al. (2016) and Marcon et al. (2017). While these can be transformed into estimators of the stable tail dependence function, they rely on stronger assumptions, i.e., they are based on data from an extreme-value distribution, and we will not consider them here.

2.2.2. Semi-parametric estimation of the stable tail dependence function

Nonparametric estimation forms a stepping stone for semi-parametric estimation (Einmahl et al., 2012; 2016; 2018). Here, we focus on the method proposed in Einmahl et al. (2018). Let $\hat{\ell}_{n,k}$ denote any initial estimator of ℓ based on $\mathbf{X}_1, \dots, \mathbf{X}_n$. Let $\mathbf{c}_1, \dots, \mathbf{c}_q \in [0, \infty)^d$, with $\mathbf{c}_m = (c_{m1}, \dots, c_{md})$ for $m = 1, \dots, q$, be q points in which we will evaluate ℓ and $\hat{\ell}_{n,k}$. Consider for $\boldsymbol{\theta} \in \Theta$ the $q \times 1$ column vectors

$$\hat{\mathbf{L}}_{n,k} := (\hat{\ell}_{n,k}(\mathbf{c}_m))_{m=1}^q, \quad L(\boldsymbol{\theta}) := (\ell(\mathbf{c}_m; \boldsymbol{\theta}))_{m=1}^q, \quad D_{n,k}(\boldsymbol{\theta}) := \hat{\mathbf{L}}_{n,k} - L(\boldsymbol{\theta}). \quad (4)$$

The points $\mathbf{c}_1, \dots, \mathbf{c}_q$ need to be chosen in such a way that the map $L: \Theta \rightarrow \mathbb{R}^q$ is one-to-one, i.e., $\boldsymbol{\theta}$ is identifiable from $\ell(\mathbf{c}_1; \boldsymbol{\theta}), \dots, \ell(\mathbf{c}_q; \boldsymbol{\theta})$. In particular, we will need to assume that $q \geq p$.

For $\boldsymbol{\theta} \in \Theta$, let $\Omega(\boldsymbol{\theta})$ be a symmetric, positive definite $q \times q$ matrix and define

$$f_{n,k}(\boldsymbol{\theta}) := D_{n,k}^T(\boldsymbol{\theta}) \Omega(\boldsymbol{\theta}) D_{n,k}(\boldsymbol{\theta}). \quad (5)$$

The continuous updating weighted least squares estimator for $\boldsymbol{\theta}_0$ is defined as

$$\hat{\boldsymbol{\theta}}_{n,k} := \arg \min_{\boldsymbol{\theta} \in \Theta} f_{n,k}(\boldsymbol{\theta}) = \arg \min_{\boldsymbol{\theta} \in \Theta} \{ D_{n,k}(\boldsymbol{\theta})^T \Omega(\boldsymbol{\theta}) D_{n,k}(\boldsymbol{\theta}) \}. \quad (6)$$

In Section 4, we will study the performance of this estimator for $\Omega(\boldsymbol{\theta}) = I_q$, the $q \times q$ identity matrix. Expression (6) then simplifies to

$$\hat{\boldsymbol{\theta}}_{n,k} = \arg \min_{\boldsymbol{\theta} \in \Theta} \sum_{m=1}^q (\hat{\ell}_{n,k}(\mathbf{c}_m) - \ell(\mathbf{c}_m; \boldsymbol{\theta}))^2.$$

Einmahl et al. (2018) show the consistency and asymptotic normality of $\hat{\boldsymbol{\theta}}_{n,k}$ under the assumption that the true parameter vector $\boldsymbol{\theta}_0$ is in the interior of Θ . In the next section we give the asymptotic distribution of $\hat{\boldsymbol{\theta}}_{n,k}$ without this restriction.

3. Inference on tail dependence parameters on the boundary of the parameter space

The asymptotic results presented in Section 3.1 build on Andrews (1999), who established the asymptotic distribution of a general extremum estimator when one or more parameters lie on the boundary of the parameter space. The methodology to obtain the asymptotic distribution of $\sqrt{k}(\hat{\boldsymbol{\theta}}_{n,k} - \boldsymbol{\theta}_0)$ can be summarized as follows: first, one shows that as $n \rightarrow \infty$, the criterion function $f_{n,k}(\boldsymbol{\theta})$ in (5) is equal to a quadratic function $q_{n,k}(\sqrt{k}(\boldsymbol{\theta} - \boldsymbol{\theta}_0))$ plus a term that does not depend on $\boldsymbol{\theta}$. If $\hat{\boldsymbol{\theta}}_{n,k}$ is consistent, then its asymptotic distribution is only affected by the part of Θ close to $\boldsymbol{\theta}_0$; equivalently, we are only concerned with the shifted parameter space $\Theta - \boldsymbol{\theta}_0$ near the origin. If $\Theta - \boldsymbol{\theta}_0$ can be approximated near the origin by a convex cone Λ , one can show that minimizing $f_{n,k}(\boldsymbol{\theta})$ over $\boldsymbol{\theta} \in \Theta$ is asymptotically equivalent to minimizing $q_{n,k}(\boldsymbol{\lambda})$ over $\boldsymbol{\lambda} \in \Lambda$. Finally, $\sqrt{k}(\hat{\boldsymbol{\theta}}_{n,k} - \boldsymbol{\theta}_0)$ converges in distribution to the argument minimizing the limit of $q_{n,k}(\boldsymbol{\lambda})$ as $n \rightarrow \infty$.

In [Section 3.2](#), we show how a closed-form expression can be obtained for the asymptotic distribution of $\sqrt{k}(\hat{\theta}_{n,k} - \theta_0)$. For $\beta \subset \Theta$, we show in [Section 3.3](#) how to test $H_0: \beta = \beta_*$ against $H_1: \beta \neq \beta_*$ when, under the null hypothesis, β_* is on the boundary of the alternative hypothesis.

We set up some notation first. Let $B_\varepsilon(\theta)$ denote an open ball centered at θ with radius ε and let $C_\varepsilon(\theta)$ denote an open cube centered at θ with sides of length 2ε . Let $\text{cl}(\Theta)$ denote the closure of Θ . A set $\Gamma \subset \mathbb{R}^p$ is said to be *locally equal* to a set $\Lambda \subset \mathbb{R}^p$ if $\Gamma \cap B_\varepsilon(\theta) = \Lambda \cap B_\varepsilon(\theta)$ for some $\varepsilon > 0$. Finally, a set $\Lambda \subset \mathbb{R}^p$ is a *cone* if $\lambda \in \Lambda$ implies $a\lambda \in \Lambda$ for all $a > 0$.

3.1. Estimation, consistency and asymptotic normality

When θ_0 is on the boundary of Θ , the map L in (4) is not defined and thus not differentiable on a neighbourhood of θ_0 . We will need the following assumption.

(A1) Θ includes a set Θ^+ such that $\Theta^+ - \theta_0$ equals the intersection of a union of orthants and an open cube $C_\varepsilon(\theta_0)$ for some $\varepsilon > 0$. Moreover, $\Theta \cap B_{\varepsilon_1}(\theta_0) \subset \Theta^+$ for some $\varepsilon_1 > 0$.

If $\Theta - \theta_0$ happens to be locally equal to a union of orthants, we can simply set $\Theta^+ = \Theta \cap C_\varepsilon(\theta_0)$. This is the case for the models considered in [Section 4](#). We will assume existence of the so-called *left/right (l/r) partial derivatives* on Θ^+ ; a formal definition is given in the appendix. The shape of Θ^+ is such that these can always be defined.

Write $\dot{L} := (\partial/\partial\theta)L(\theta) \in \mathbb{R}^{q \times p}$ for $\theta \in \Theta^+$, where \dot{L} denotes the matrix of l/r partial derivatives. Let $\lambda_1(\theta) > 0$ denote the smallest eigenvalue of $\Omega(\theta)$.

Theorem 3.1 (Existence, uniqueness and consistency). *Let $\mathbf{c}_1, \dots, \mathbf{c}_q \in [0, \infty)^d$ be $q \geq p$ points such that the map $L: \theta \mapsto (\ell(\mathbf{c}_m; \theta))_{m=1}^q$ is a homeomorphism. Let $\theta_0 \in \text{cl}(\Theta)$ and assume that (A1) holds, that each element of $L(\theta)$ has continuous l/r partial derivatives of order two on Θ^+ , that $\dot{L}(\theta_0)$ is of full rank, that $\Omega: \Theta \rightarrow \mathbb{R}^{q \times q}$ has continuous l/r partial derivatives on Θ^+ and that $\inf_{\theta \in \Theta} \lambda_1(\theta) > 0$. Finally assume, for $m = 1, \dots, q$,*

$$\hat{\ell}_{n,k}(\mathbf{c}_m) \xrightarrow{p} \ell(\mathbf{c}_m; \theta_0), \quad \text{as } n \rightarrow \infty. \quad (7)$$

Then with probability tending to one, the minimizer $\hat{\theta}_{n,k}$ in (6) exists and is unique. Moreover,

$$\hat{\theta}_{n,k} \xrightarrow{p} \theta_0, \quad \text{as } n \rightarrow \infty.$$

We omit the proof of this theorem since it is directly obtained by replacing $B_\varepsilon(\theta_0)$ by Θ^+ in the proof of [Einmahl et al. \(2018, Theorem 1\)](#).

When $f_{n,k}$ is not defined on a neighbourhood of θ_0 , but there exists a set Θ^+ which satisfies (A1), a Taylor expansion of $f_{n,k}(\theta)$ around $f_{n,k}(\theta_0)$ holds ([Andrews, 1999, Theorem 6](#)). For each $\theta \in \Theta^+$, we have

$$f_{n,k}(\theta) = f_{n,k}(\theta_0) + Df_{n,k}(\theta_0)^T(\theta - \theta_0) + (\theta - \theta_0)^T D^2 f_{n,k}(\theta_0)(\theta - \theta_0)/2 + R_{n,k}(\theta),$$

where $Df_{n,k}$ and $D^2 f_{n,k}$ are based on l/r partial derivatives and $R_{n,k}(\theta)$ is the remainder term. We suppress dependence of Ω , \dot{L} and $D_{n,k}$ on θ_0 for ease of notation. From [Einmahl et al. \(2018, Proof of Theorems 3.2.1 and 3.2.2\)](#) we know that

$$Df_{n,k}(\theta_0) = -2D_{n,k}^T \Omega \dot{L} + o_p(1), \quad D^2 f_{n,k}(\theta_0) = 2\dot{L}^T \Omega \dot{L} + o_p(1),$$

so the quadratic expansion above is equal to

$$f_{n,k}(\theta) = f_{n,k}(\theta_0) - 2D_{n,k}^T \Omega \dot{L}(\theta - \theta_0) + (\theta - \theta_0)^T \dot{L}^T \Omega \dot{L}(\theta - \theta_0) + R_{n,k}(\theta).$$

Define

$$J := \dot{L}^T \Omega \dot{L} \in \mathbb{R}^{p \times p}, \quad \mathbf{Y}_{n,k} := \sqrt{k} J^{-1} \dot{L}^T \Omega D_{n,k} \in \mathbb{R}^p,$$

and

$$q_{n,k}(\lambda) = (\lambda - \mathbf{Y}_{n,k})^T J (\lambda - \mathbf{Y}_{n,k}).$$

Then $f_{n,k}(\theta)$ can be written as

$$f_{n,k}(\theta) = f_{n,k}(\theta_0) - k^{-1} \mathbf{Y}_{n,k}^T J \mathbf{Y}_{n,k} + k^{-1} q_{n,k}(\sqrt{k}(\theta - \theta_0)) + R_{n,k}(\theta).$$

We see that $f_{n,k}(\theta)$ is equal to a quadratic function plus a term that does not depend on θ plus $R_{n,k}(\theta)$. In [Andrews \(2002, Lemma 3\)](#), it is shown that under the assumptions of [Theorem 3.1](#) and (A3) below, the remainder term $R_{n,k}(\theta)$ is sufficiently small so that minimizing the quadratic approximation to $f_{n,k}(\theta)$ is equivalent to minimizing $q_{n,k}(\sqrt{k}(\theta - \theta_0))$. Now, assume that

(A2) $\Theta - \theta_0$ is locally equal to a convex cone $\Lambda \subset \mathbb{R}^p$.

Then

$$\inf_{\theta \in \Theta} q_{n,k}(\sqrt{k}(\theta - \theta_0)) = \inf_{\lambda \in \Lambda} q_{n,k}(\lambda) + o_p(1),$$

for some cone Λ . Finally, assume that

$$(A3) \quad \sqrt{k}D_{n,k}(\theta_0) \xrightarrow{d} \mathbf{D} \sim \mathcal{N}_q(0, \Sigma(\theta_0)) \text{ as } n \rightarrow \infty, \text{ for some covariance matrix } \Sigma(\theta_0).$$

Assumptions (A3) and (7) hold for the nonparametric estimators presented in Section 2.2.1 under some general regularity conditions. The matrix Σ can be expressed in terms of ℓ and its first-order partial derivatives. We have

$$\mathbf{Y}_{n,k} \xrightarrow{d} \mathbf{Y} := J^{-1} \dot{L}^T \Omega \mathbf{D}, \quad q_{n,k}(\lambda) \xrightarrow{d} q(\lambda) := (\lambda - \mathbf{Y})^T J (\lambda - \mathbf{Y}),$$

for all $\lambda \in \mathbb{R}^p$.

Theorem 3.2 (Asymptotic normality). *If all of the above assumptions hold,*

$$\sqrt{k}(\hat{\theta}_{n,k} - \theta_0) \xrightarrow{d} \hat{\lambda} = \arg \min_{\lambda \in \Lambda} q(\lambda), \quad \text{as } n \rightarrow \infty.$$

Here $\hat{\lambda}$ can be interpreted as the projection of \mathbf{Y} on Λ with respect to the norm $\|y\|_J := y^T J y$. This theorem is a special case of Andrews (1999, Theorem 3). In the appendix, we verify that our assumptions are sufficient. Note that if Θ includes a neighbourhood of θ_0 , then $\Lambda = \mathbb{R}^p$ and we find the same distribution as in Einmahl et al. (2018, Theorem 2) since $\hat{\lambda} = \mathbf{Y}$.

3.2. Simplifying the asymptotic distribution

The goal of this section is to simplify the asymptotic distribution of $\sqrt{k}(\hat{\theta}_{n,k} - \theta_0)$ and to give conditions under which (part of) $\hat{\lambda}$ has a closed-form expression. Let $\mathbf{G} := \dot{L}^T \Omega \mathbf{D}$, i.e., $\mathbf{Y} = J^{-1} \mathbf{G}$. We start by partitioning the vector $\theta_0 \in \mathbb{R}^p$ into two subvectors, $\beta_0 \in \mathbb{R}^c$ and $\delta_0 \in \mathbb{R}^{p-c}$ for $c \in \{1, \dots, p\}$, where δ_0 consists of all parameters that are in the interior of the parameter space. We partition $\theta_{n,k}$, \mathbf{Y} , J , \mathbf{G} and λ accordingly:

$$\theta_{n,k} = \begin{pmatrix} \beta_{n,k} \\ \delta_{n,k} \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} \mathbf{Y}_\beta \\ \mathbf{Y}_\delta \end{pmatrix}, \quad J = \begin{pmatrix} J_\beta & J_{\beta\delta} \\ J_{\delta\beta} & J_\delta \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} \mathbf{G}_\beta \\ \mathbf{G}_\delta \end{pmatrix}, \quad \lambda = \begin{pmatrix} \lambda_\beta \\ \lambda_\delta \end{pmatrix}.$$

Let I_c denote the $c \times c$ identity matrix. For $H := (I_c : \mathbf{0}) \in \mathbb{R}^{c \times p}$, define

$$q_\beta(\lambda_\beta) := (\lambda_\beta - \mathbf{Y}_\beta)^T (HJ^{-1}H^T)^{-1} (\lambda_\beta - \mathbf{Y}_\beta).$$

Suppose that

$$(A4) \quad \text{The cone } \Lambda \text{ of assumption (A2) is equal to the product set } \Lambda_\beta \times \mathbb{R}^{p-c}, \text{ where } \Lambda_\beta \subset \mathbb{R}^c \text{ is a cone.}$$

Corollary 3.3. *If all of the above assumptions hold, then*

$$\begin{aligned} \sqrt{k}(\hat{\beta}_{n,k} - \beta_0) &\xrightarrow{d} \hat{\lambda}_\beta := \arg \min_{\lambda_\beta \in \Lambda_\beta} q_\beta(\lambda_\beta), \\ \sqrt{k}(\hat{\delta}_{n,k} - \delta_0) &\xrightarrow{d} J_\delta^{-1} \mathbf{G}_\delta - J_\delta^{-1} J_{\delta\beta} \hat{\lambda}_\beta. \end{aligned}$$

When Λ_β is defined by equality and/or inequality constraints, a closed-form expression for $\hat{\lambda}_\beta$ can be computed; we give some examples that are relevant for Section 4. For a more formal solution, see Andrews (1999, Theorem 5).

Example 3.1 ($c = 1$). Suppose that $\Theta = [0, 1]^p$. If $\beta_0 = 0$, then $\Lambda_\beta = [0, \infty)$ and $\hat{\lambda}_\beta = \max(Y_\beta, 0)$. If $\beta_0 = 1$, then $\Lambda_\beta = (-\infty, 0]$ and $\hat{\lambda}_\beta = \min(Y_\beta, 0)$.

Example 3.2 ($c = 2$). Suppose that $\Theta = [0, 1]^p$. If $\beta_0 = (1, 1)$, then $\Lambda_\beta = (-\infty, 0]^2$. Let $\rho := HJ^{-1}H^T$. Then

$$\begin{aligned} \hat{\lambda}_\beta &:= \mathbb{1}\{Y_{\beta 1} < 0, Y_{\beta 2} < 0\} \mathbf{Y}_\beta + \mathbb{1}\{Y_{\beta 1} - \rho_{21}Y_{\beta 2} < 0, Y_{\beta 2} \geq 0\} (Y_{\beta 1} - \rho_{21}Y_{\beta 2}, 0)^T \\ &\quad + \mathbb{1}\{Y_{\beta 1} \geq 0, Y_{\beta 2} - \rho_{12}Y_{\beta 1} < 0\} (0, Y_{\beta 2} - \rho_{12}Y_{\beta 1})^T. \end{aligned}$$

If $\beta_0 = (0, 0)$, $\beta_0 = (0, 1)$ or $\beta_0 = (1, 0)$, similar expressions can be obtained by reversing the first and/or the second inequalities in each of the indicator functions above.

3.3. Hypothesis testing for tail dependence parameters on the boundary of the parameter space

We are interested in constructing hypothesis tests for parameter values that, under the null hypothesis, are on the boundary of the alternative hypothesis. We propose two test statistics, whose asymptotic distribution follows from the results in Andrews (2001).

Assume that there are no nuisance parameters on the boundary, i.e., all components of θ_0 that lie on the boundary are part of the null hypothesis. We are interested in testing

$$H_0 : \beta = \beta_* \quad \text{vs} \quad H_1 : \beta \neq \beta_*, \quad \beta_* \in \mathbb{R}^c.$$

Let $\Theta_0 := \{\theta \in \Theta : \theta = (\beta_*, \delta) \text{ for some } \delta \in \mathbb{R}^{p-c}\}$ denote the restricted parameter space. Assume that

(A5) For all $\theta \in \Theta_0$, Θ is a product set with respect to (β, δ) local to θ . That is, for all $\theta \in \Theta_0$,

$$\Theta \cap B_\varepsilon(\theta) = (B \times \Delta) \cap B_\varepsilon(\theta) \quad \text{for some } B \subset \mathbb{R}^c, \Delta \subset \mathbb{R}^{p-c} \text{ and } \varepsilon > 0.$$

Define $\hat{\theta}_{n,k}^{(0)} := \arg \min_{\theta \in \Theta_0} f_{n,k}(\theta)$. A deviance test statistic can be defined as

$$T_{n,k}^{(1)} := k \left(f_{n,k}(\hat{\theta}_{n,k}^{(0)}) - f_{n,k}(\hat{\theta}_{n,k}) \right).$$

Corollary 3.4. Suppose $\theta_0 \in \Theta_0$ and all previous assumptions hold. Then

$$T_{n,k}^{(1)} \xrightarrow{d} \hat{\lambda}_\beta^T (HJ^{-1}H^T)^{-1} \hat{\lambda}_\beta.$$

Recall that $G = \dot{L}^T \Omega \dot{D}$ and let \mathcal{J} denote its covariance matrix, i.e., $G \sim \mathcal{N}_p(\mathbf{0}, \mathcal{J})$ with $\mathcal{J} = \dot{L}^T \Omega \Sigma \Omega \dot{L} \in \mathbb{R}^{p \times p}$. Note that $\mathcal{J} = J$ if and only if $\Omega = \Sigma^{-1}$. If $\Lambda_\beta = \mathbb{R}^c$ (no parameters on the boundary) and $\mathcal{J} = J$, then

$$T_{n,k}^{(1)} \xrightarrow{d} Y_\beta^T (HJ^{-1}H^T)^{-1} Y_\beta \sim \chi_c^2,$$

where χ_c^2 denotes a chi-squared random variable with c degrees of freedom.

A Wald-type test statistic can be based on the quadratic form in $\hat{\beta}_{n,k} - \beta_*$. Let $\hat{V}_n^{-1} := (HJ_n^{-1}H^T)^{-1} \in \mathbb{R}^{c \times c}$ denote a weight matrix where $J_n := J(\hat{\theta}_{n,k})$. Define

$$T_{n,k}^{(2)} := k \left(\hat{\beta}_{n,k} - \beta_* \right)^T \hat{V}_n^{-1} \left(\hat{\beta}_{n,k} - \beta_* \right).$$

Corollary 3.5. Suppose $\theta_0 \in \Theta_0$ and all of the above assumptions hold. Then

$$T_{n,k}^{(2)} \xrightarrow{d} \hat{\lambda}_\beta^T (HJ^{-1}H^T)^{-1} \hat{\lambda}_\beta.$$

Corollary 3.4 is a special case of Andrews (2001, Theorem 4c) and Corollary 3.5 is a special case of Andrews (2001, Theorem 6d); their proofs are direct and thus omitted.

4. Simulation studies

We conduct simulation experiments where we simulate 1000 samples of size $n = 5000$ from a parametric model and we assess the quality of the weighted least squares estimator in terms of its root mean squared error (RMSE) for settings where one or more of the parameters are on the boundary of the parameter space. We take $k \in \{25, 50, \dots, 300\}$ and we use the empirical and the beta tail dependence function as initial estimators. Next, we study the performance of the two test statistics introduced in Section 3.3 in terms of empirical level and power for $k \in \{25, 50, 75, 100\}$, as we observe that higher k leads to a steadily growing bias that quickly deteriorates the performance of the hypothesis test. In all experiments, we take $\Omega = I_q$, the $q \times q$ identity matrix, because using an optimal weight matrix has a very minor effect on the quality of estimation while severely slowing down the estimation procedure. Moreover, inverting Σ may be hindered by numerical problems in the case of a max-linear model (Einmahl et al., 2018).

4.1. Brown–Resnick model

We simulate data from a Brown–Resnick model with $\alpha = 2$ and $\rho = 1$ on a regular 3×2 grid ($d = 6$) and on a regular 4×4 grid ($d = 16$). As in Einmahl et al. (2018), we let $\mathbf{c}_m \in \{0, 1\}^d$ such that exactly two components of \mathbf{c}_m are equal to one (meaning that we consider a pairwise estimator) and we focus on pairs of neighbouring locations only, i.e., locations that are at most a distance of $\sqrt{2}$ apart. This leads to $q = 11$ and $q = 42$ respectively.

Fig. 1 shows the RMSE of the parameter estimates based on the empirical tail dependence function (solid lines) and the beta tail dependence function (dashed lines). We see that the empirical tail dependence function outperforms the beta tail dependence function for α , while the opposite is true for ρ , although differences are very minor. For small values of k , the RMSE is lower for $d = 16$ than for $d = 6$.

Table 1

Empirical level of $T_{n,k}^{(2)}$ based on the empirical and the beta tail dependence function for a significance level of 0.05.

	$k = 25$		$k = 50$		$k = 75$		$k = 100$	
	emp	beta	emp	beta	emp	beta	emp	beta
$d = 6$	6.0	4.3	5.1	5.0	7.0	5.5	8.8	8.0
$d = 16$	6.7	10.3	8.3	10.2	9.8	12.4	13.2	16.0

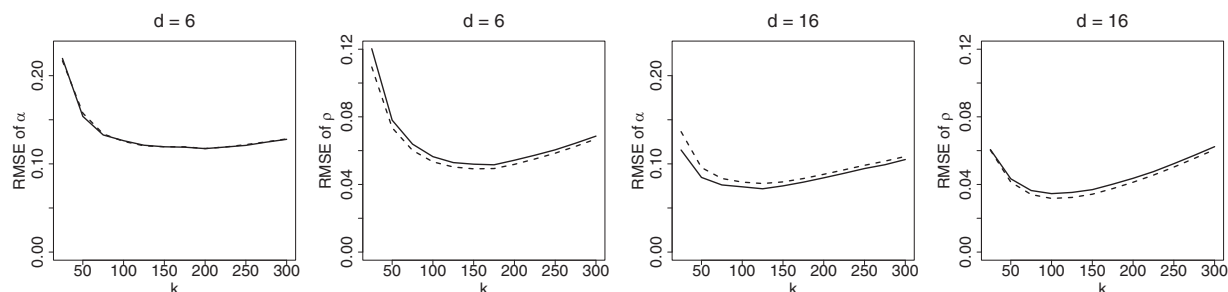


Fig. 1. RMSE for parameter estimators of the Brown-Resnick model based on the empirical tail dependence function (solid lines) and the beta tail dependence function (dashed lines) for $(\alpha, \rho) = (2, 1)$; $d = 6$ (left) and $d = 16$ (right).

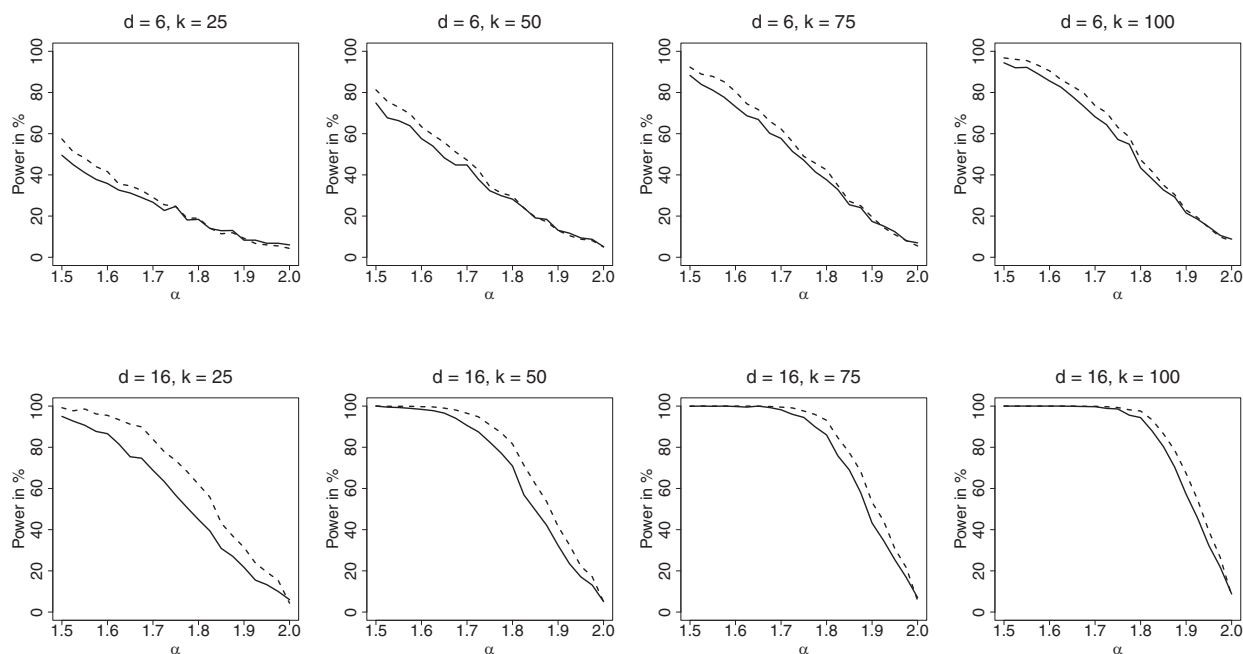


Fig. 2. Empirical power of $T_{n,k}^{(2)}$ based on the empirical tail dependence function (solid lines) and the beta tail dependence function (dashed lines) for $k \in \{25, 50, 75, 100\}$; $d = 6$ (top) and $d = 16$ (bottom).

Next, we study the empirical level and power of the test $H_0: \alpha = 2$. Assumption (A4) holds with $\Lambda = (-\infty, 0] \times \mathbb{R}$. Table 1 shows the empirical level of the test statistic $T_{n,k}^{(2)}$ based on a significance level of $\alpha = 0.05$. We do not display $T_{n,k}^{(1)}$ as its behaviour is identical to that of $T_{n,k}^{(2)}$. Surprisingly, the empirical tail dependence function is preferred for $d = 16$ while the beta tail dependence function performs best for $d = 6$. In general, we can conclude that low values of k are to be preferred and that the test performs better for lower dimension. Fig. 2 shows the empirical power of $T_{n,k}^{(2)}$ as a function of $\alpha \in [1.5, 2)$. The power increases both with the dimension and with k .

Table 2

Empirical level of $T_{n,k}^{(1)}$ and $T_{n,k}^{(2)}$ for models 1–3 (M1–M3) based on the empirical and the beta tail dependence function for a significance level of 0.05.

$T_{n,k}^{(1)}$	$k = 25$		$k = 50$		$k = 75$		$k = 100$	
	emp	beta	emp	beta	emp	beta	emp	beta
M1	2.9	2.1	6.1	4.2	5.9	5.3	8.0	6.4
M2	0.3	2.3	8.7	5.0	12.4	12.6	40.3	27.7
M3	3.3	7.9	7.9	9.7	6.5	9.8	9.8	10.0
$T_{n,k}^{(2)}$	$k = 25$		$k = 50$		$k = 75$		$k = 100$	
	emp	beta	emp	beta	emp	beta	emp	beta
M1	3.0	2.9	6.2	5.0	6.1	6.1	8.5	7.1
M2	0.3	2.4	9.2	5.3	12.5	13.0	41.3	28.3
M3	5.4	9.9	10.3	10.2	8.4	10.7	10.7	11.1

4.2. Max-linear model

The constants $\mathbf{c}_1, \dots, \mathbf{c}_q$ need to be chosen such that the max-linear model is identifiable. In Einmahl et al. (2018) it was observed that taking extremal coefficients for \mathbf{c}_m , i.e., $\mathbf{c}_m \in \{0, 1\}^d$ which at least two non-zero elements, is not enough for identifiability. Instead, one needs to choose \mathbf{c}_m such that its non-zero elements are unequal. For some theoretical considerations, see also Einmahl et al. (2018, Appendix B). Simulation experiments showed that in practice, taking a large grid of values (meaning that $q \gg p$) on $[0, 1]^d$ will lead to good estimators in terms of RMSE. Hence, in all following simulation experiments, we choose $\mathbf{c}_1, \dots, \mathbf{c}_q$ such that

$$\mathbf{c}_m \in \{0, 0.01, 0.1, 0.2, \dots, 0.8, 0.9, 0.99, 1\}^2, \quad m \in \{1, \dots, q\}.$$

4.2.1. Testing the structure of a max-linear model

We consider three different scenarios:

Model 1 Let $r = 2$ and $d = 3$, so that $\Theta = [0, 1]^3$. Let $\theta_0 = (b_{11}, b_{21}, b_{31}) = (1, 0.7, 0.2)$. We test $H_0 : b_{11} = 1$. Assumption (A4) holds with $\Lambda = (-\infty, 0] \times \mathbb{R}^2$.

Model 2 Let $r = 2$ and $d = 3$, so that $\Theta = [0, 1]^3$. Let $\theta_0 = (b_{11}, b_{21}, b_{31}) = (1, 0.7, 0)$. We test $H_0 : (b_{11}, b_{31}) = (1, 0)$. Assumption (A4) holds with $\Lambda = (-\infty, 0] \times [0, \infty) \times \mathbb{R}$.

Model 3 Let $r = 3$ and $d = 2$, so that $\Theta = [0, 1]^4$. Let $\theta_0 = (b_{11}, b_{21}, b_{12}, b_{22}) = (0, 0.8, 0.6, 0)$. We test $H_0 : (b_{11}, b_{22}) = (0, 0)$. Assumption (A4) holds with $\Lambda = [0, \infty) \times \mathbb{R}^2 \times [0, \infty)$. If the null hypothesis cannot be rejected, it means that a Marshall–Olkin model suffices.

Fig. 3 shows the RMSE of the parameter estimates of the three max-linear models explained above, based on the empirical tail dependence function (solid lines) and the beta tail dependence function (dashed lines). We see again that these two initial estimators lead to similar results. Parameters whose true values are on the boundary are better estimated for low k , while higher values of k are preferred for parameters whose true values are in the interior of the parameter space.

Table 2 shows the empirical level of the test statistics $T_{n,k}^{(1)}$ and $T_{n,k}^{(2)}$ for $k \in \{25, 50, 75, 100\}$, using a significance level of 0.05. The tests perform well for model 1 in general, while low values of k are necessary for models 2 and 3. The beta tail dependence function outperforms the empirical tail dependence function for model 2, while the opposite is the case for model 3. Overall, test statistic $T_{n,k}^{(1)}$ performs somewhat better than $T_{n,k}^{(2)}$.

To assess the power of the two test statistics, we consider the models described above, but we let

$$\begin{aligned} b_{11} &\in \{0.8, 0.85, \dots, 1\} \quad \text{for model 1,} \\ b_{11} &\in \{0.8, 0.85, \dots, 1\}, \quad b_{31} \in \{0, 0.05, \dots, 0.2\} \quad \text{for model 2, and} \\ b_{11}, b_{22} &\in \{0, 0.05, \dots, 0.2\} \quad \text{for model 3.} \end{aligned}$$

Fig. 4 shows the empirical power of $T_{n,k}^{(1)}$ for models 1–3 for the beta tail dependence function (results for the empirical tail dependence function are very similar). The most striking result of the lower-left panel is that the graph is not symmetric, i.e., the power is higher when b_{22} is near zero (and b_{11} is not) than when b_{11} is near zero (and b_{22} is not).

4.2.2. Testing the number of factors in a max-linear model

We studied the performance of the test statistics in case we wanted to identify a specific submodel of the max-linear model. We would also like to investigate the capability of the test statistics to correctly retrieve the true number of factors, i.e., for $s \in \{1, \dots, r\}$, we wish to test the hypothesis $H_0 : (b_{1s}, \dots, b_{ds}) = \mathbf{0}$. However, as mentioned in Section 2.1, the max-linear model is only defined for parameter values with $\sum_{j=1}^d b_{js} > 0$ for all $s \in \{1, \dots, d\}$; when this condition is not met, there are zeroes on the diagonal of $\Sigma(\theta_0)$, making computation of $\hat{\lambda}_\beta$ impossible. We solve this problem by computing

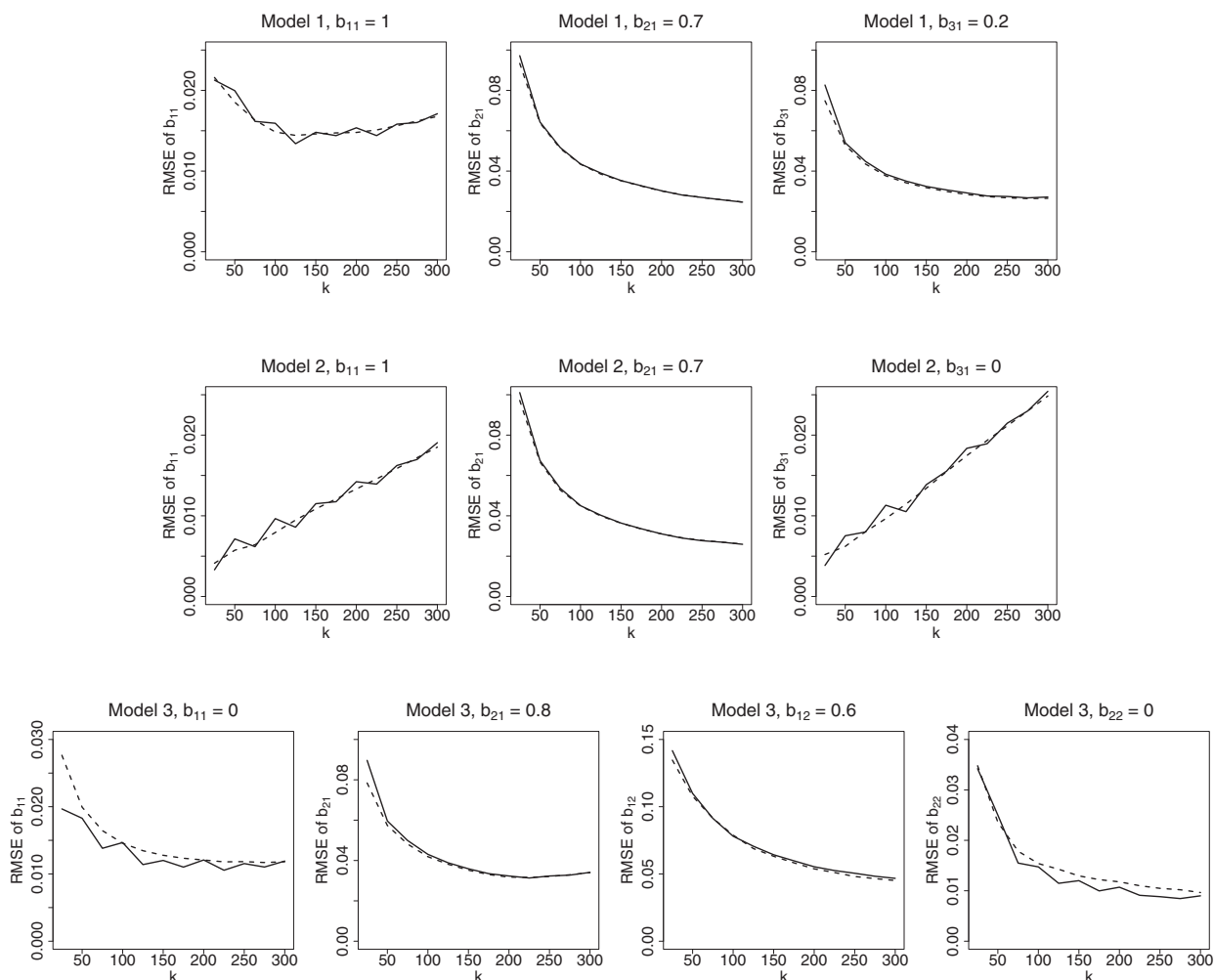


Fig. 3. RMSE for models 1–3 based on the empirical tail dependence function (solid lines), and the beta tail dependence function (dashed lines).

Table 3

Empirical level of $T_{n,k}^{(2)}$ based on the empirical and the beta tail dependence function for a significance level of 0.05.

	$k = 25$		$k = 50$		$k = 75$		$k = 100$	
	emp	beta	emp	beta	emp	beta	emp	beta
$T_{n,k}^{(2)}$	4.6	23.9	2.7	24.9	2.4	23.8	2.5	20.0

the asymptotic distribution of the test statistics using $\hat{\theta}_{n,k}$ rather than θ_0 . We consider a model with $d = 2$ and $r = 3$, with parameter vector $\theta_0 = (b_{11}, b_{21}, b_{12}, b_{22}) = (0.8, 0.6, 0.2, 0.4)$, i.e., the third column of B contains only zeroes and the model has effectively two factors. We estimate a three-factor model and we test $H_0 : (b_{13}, b_{23}) = (0, 0)$.

Fig. 5 shows the RMSE of the parameter estimates based on the empirical tail dependence function (solid lines) and the beta tail dependence function (dashed lines). We see that, contrary to previous experiments, the beta tail dependence has much higher RMSE than the empirical tail dependence function: oversmoothing leads to a large bias for (b_{13}, b_{23}) (and hence for the other parameters as well).

Table 3 shows the empirical level of the test statistic $T_{n,k}^{(2)}$ using a significance level of 0.05. We included the results based on the beta tail dependence function for completeness: because it tends to overestimate (b_{13}, b_{23}) , it rejects the null hypothesis far too often. Test statistic $T_{n,k}^{(1)}$ is not presented because for any k and any initial estimator, it has an empirical level of 0. Even when $|\hat{\theta}_{n,k} - \hat{\theta}_{n,k}^{(0)}|$ is large, $|f_{n,k}(\hat{\theta}_{n,k}) - f_{n,k}(\hat{\theta}_{n,k}^{(0)})|$ is small and hence the deviance-type test is not adapted to this type of model. Finally, Fig. 6 shows the empirical power of $T_{n,k}^{(2)}$ for $b_{11}, b_{22} \in \{0, 0.05, \dots, 0.2\}$ for two values of k .

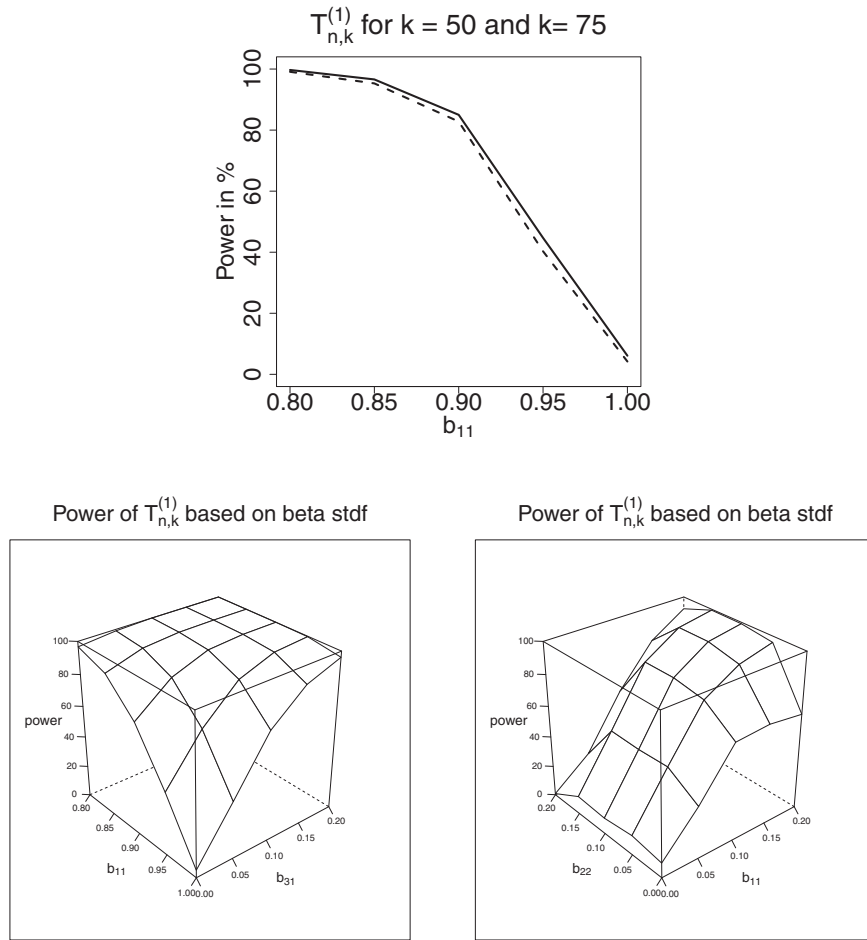


Fig. 4. Top: empirical power in % for model 1 and $k = 50$ (dashed line), $k = 75$ (solid line), based on the empirical tail dependence function. Bottom: empirical power in % for model 2 (left) and model 3 (right) and $k = 50$ based on the beta tail dependence function.

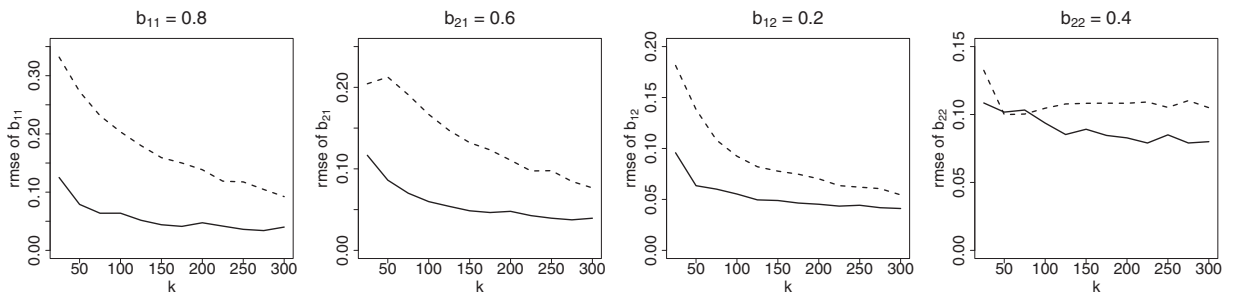


Fig. 5. RMSE based on the empirical tail dependence function (solid lines) and the beta tail dependence function (dashed lines).

We remark that the power is similar when one parameter is far from its boundary and when both parameters are far from their boundary.

5. Application to stock market indices

Consider two major European stock market indices, the German DAX and the French CAC40. We take the daily negative log-returns of the prices of these two indices from <https://finance.yahoo.com> for the period of January 1st, 1997 to December 31st, 2017. We remove all dates for which at least one of the two series has missing values, ending up with a sample of size $n = 5313$. Fig. 7 shows the time series plots of log-returns and the dependence structure for the returns standardized to unit Pareto margins, plotted on the exponential scale. The time series plots look stationary: we perform a Augmented

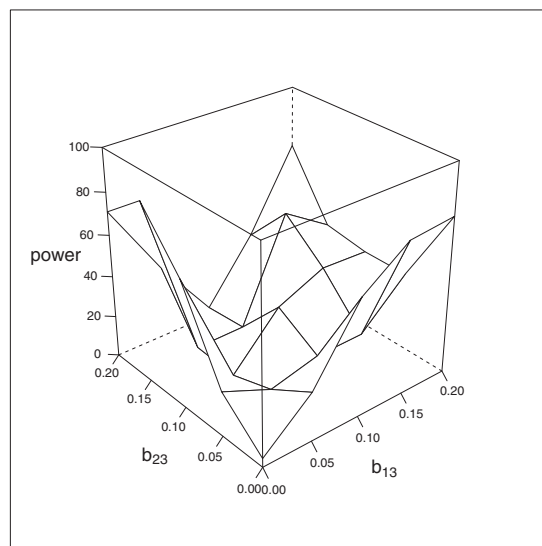
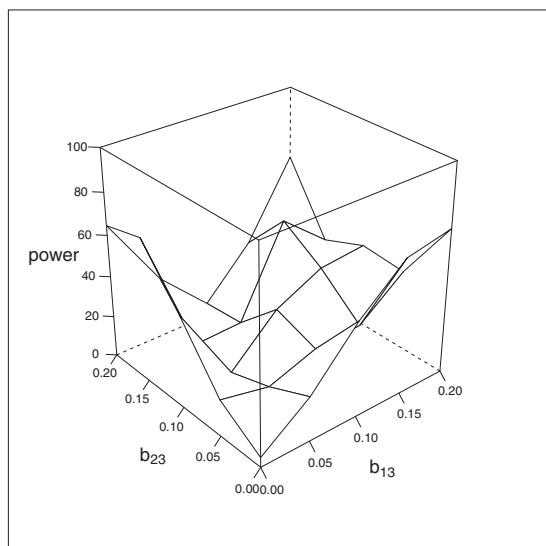
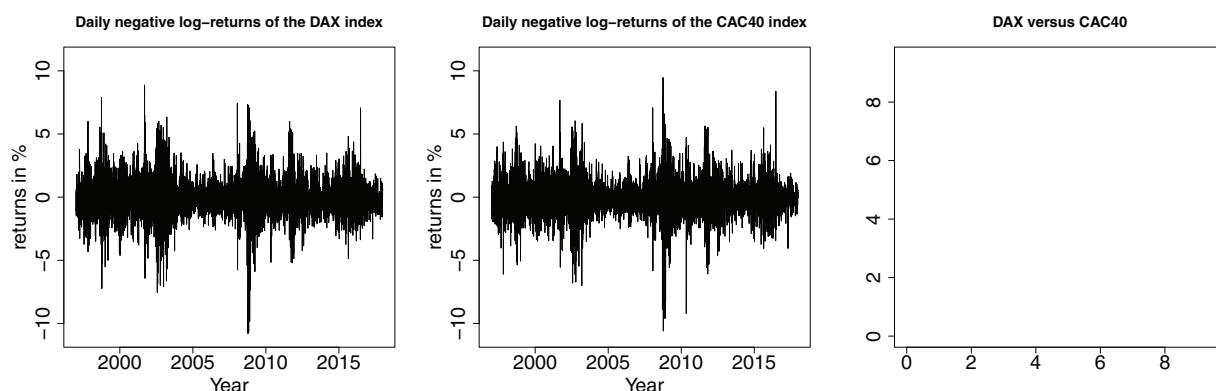
Power of $T_{n,k}^{(2)}$ based on empirical stdf, $k = 50$ Power of $T_{n,k}^{(2)}$ based on empirical stdf, $k = 75$ Fig. 6. Empirical power in % of $T_{n,k}^{(2)}$, based on the empirical tail dependence function for $k = 50$ (left) and $k = 75$ (right).

Fig. 7. Left and middle: time series plots of daily negative log-returns of the DAX and the CAC40; right: scatterplot of daily negative log-returns of the DAX versus the CAC40 on the unit Pareto scale, plotted on the exponential scale.

Table 4

Parameter matrix for a bivariate max-linear model with $r = 4$ factors for $k = 40$; standard errors are in parentheses.

DAX	0.41 (0.11)	0.14 (0.05)	0.43 (0.11)	0.03 (0.05)
CAC40	0.43 (0.12)	0.44 (0.10)	0.13 (0.05)	0.00 (0.01)

Dickey–Fuller test, which gives p -values below 0.01 for both series, and a KPSS test, giving p -values above 0.1 for both series. Hence, the assumption of stationarity is reasonable. Let u_j denote the 0.95% quantile of X_{1j}, \dots, X_{nj} for $j \in \{1, \dots, d\}$. We fit a generalized Pareto distribution (GPD) to $X_{ij} - u_j \mid X_{ij} > u_j$ and obtain the parameter estimates $\hat{\sigma}_1 = 1.12$ (0.11), $\hat{\gamma}_1 = 0.03$ (0.08), $\hat{\sigma}_2 = 1.10$ (0.10) and $\hat{\gamma}_2 = 0.02$ (0.07).

Table 4 shows the estimated parameter matrix $B = (b_{jt})_{j,t}$ for $j = 1, 2$ and $t = 1, 2, 3, 4$ obtained by fitting a four-factor max-linear model to our data based on the empirical tail dependence function. Values of $k \in \{25, 30, \dots, 70, 75\}$ were considered: we chose $k = 40$ because parameter estimates are stable around this value. Standard errors were calculated using the asymptotic variance matrix of the estimator. The results in Table 4 suggest to test $H_0 : (b_{14}, b_{24}) = (0, 0)$; the value of the test statistic $T_{n,k}^{(2)}$ is 0.04. Comparing with a critical value of 17.4 based on a significance level of 0.05, we find that we cannot reject the null hypothesis. Critical values are calculated by simulation from the asymptotic distribution of the test statistics, which is given in Corollaries 3.4 and 3.5. We do not consider $T_{n,k}^{(1)}$ or the beta tail dependence function because of their bad performance (see Section 4.2.2).

Table 5

Parameter matrix for a bivariate max-linear model with $r = 3$ factors for $k = 40$; standard errors are in parentheses.

DAX	0.41 (0.11)	0.14 (0.06)	0.46 (0.09)
CAC40	0.43 (0.12)	0.44 (0.10)	0.13 (0.05)

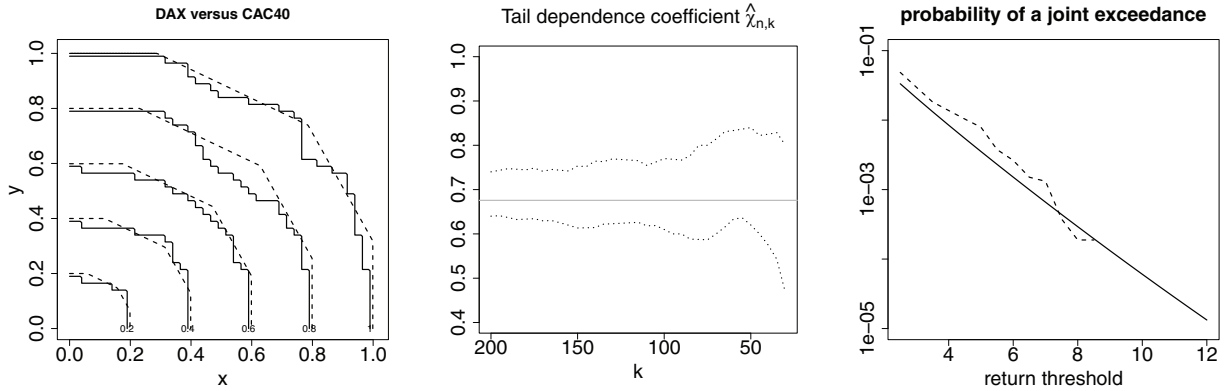


Fig. 8. Left: level sets $\{(x_1, x_2) : \ell(x_1, x_2) = c\}$ for $c \in \{0.2, 0.4, 0.8, 1\}$ based on the empirical tail dependence function (solid lines) and on the fitted max-linear model (dashed lines); middle: nonparametric (black dots) and model-based (red dots) estimates of the tail dependence coefficient $\hat{\chi}_{n,k}$ for $k \in \{200, 195, \dots, 30\}$; right: probability of a joint exceedance based on the fitted max-linear model (solid line) and on the empirical excesses (dashed line). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Table 5 shows the estimated parameter matrix $B = (b_{jt})_{j,t}$ for $j = 1, 2$ and $t = 1, 2, 3$ for a three-factor model. We test if a Marshall–Olkin model suffices, i.e., $H_0 : (b_{12}, b_{23}) = (0, 0)$. The value of the test statistic $T_{n,k}^{(2)}$ is 75.9. Comparing to a critical value of 8.03, we reject the null hypothesis of a Marshall–Olkin model.

Fig. 8 shows two goodness-of-fit measures and the estimated probability of a joint exceedances of the two stocks. On the left, we plotted the level sets $\{(x_1, x_2) : \ell(x_1, x_2) = c\}$ for $c \in \{0.2, 0.4, 0.8, 1\}$ based on the empirical tail dependence function (solid lines) and on the fitted max-linear model (dashed lines).

A common summary measure of dependence is the tail dependence coefficient,

$$\chi(q) := \lim_{q \uparrow 1} \mathbb{P}[F_1(X_{11}) < q \mid F_2(X_{12}) < q] = 2 - \ell(1, 1),$$

see (2). Fig. 8 (middle) shows nonparametric estimates of the tail dependence coefficient (black dots) $\tilde{\chi}_{n,k} = 2 - \tilde{\ell}_{n,k}(1, 1)$ and its model-based counterpart (red dots) for decreasing k . The horizontal grey line corresponds to the value $k = 40$ that was used for the parameter estimators and the tests, corresponding to $\hat{\chi}_{n,k} \approx 0.68$, which is equal to the model-based χ . The dotted lines correspond to 95% pointwise bootstrap confidence intervals.

The probability of a joint excess $\bar{F}(x_1, x_2) = \mathbb{P}[X_{11} > x_1, X_{12} > x_2]$ can be approximated for large (x_1, x_2) by

$$\bar{F}(x_1, x_2) = x_1^* + x_2^* - 1 + \exp\{-\ell(x_1^*, x_2^*)\} \quad \text{for } x_j^* = 1 - F_j(x_j), \quad j = 1, 2.$$

Let $\bar{H}(\cdot; \gamma_j, \sigma_j)$ denote the survival function of a GPD with shape γ_j and scale σ_j . For $x_1 > u_1$ and $x_2 > u_2$, we can estimate $\bar{F}(x_1, x_2)$ by

$$\hat{\bar{F}}(x_1, x_2) = \hat{x}_1^* + \hat{x}_2^* - 1 + \exp\left\{-\ell(\hat{x}_1^*, \hat{x}_2^*; \hat{\theta}_{n,k})\right\},$$

where $\ell(\cdot; \hat{\theta}_{n,k})$ denotes the max-linear stable tail dependence function based on the fitted parameter estimates and $\hat{x}_j^* = 0.05 \times \bar{H}(x_j - u_j; \hat{\gamma}_j, \hat{\sigma}_j)$. The right-hand plot of Fig. 7 shows the probabilities $\hat{\bar{F}}(x, x)$ on the log-scale for $x \in \{2.5, 3, \dots, 12\}$ (solid line). These can be compared to the empirical excess probabilities $n^{-1} \sum_{i=1}^n \mathbb{1}\{X_{1i} > x, X_{2i} > x\}$ (dashed line), which are zero for $x > 8.5$.

6. Discussion

We proposed two test statistics for parameters that, under the null hypothesis, are on the boundary of the alternative hypothesis. These test statistics are convenient because their asymptotic distribution has an explicit expression, and have been shown to perform well in simulation studies. In practice, their asymptotic distribution becomes cumbersome when testing for a higher dimensional ($c > 4$) parameter vector. A possible solution might be to consider a multiple testing procedure with a Bonferroni-type of correction.

We applied the test statistics to the max-linear model and the Brown–Resnick model, but the results are generic and could be used on any multivariate extreme-value model where submodels appear at boundary values or where the number of “factors” is of interest. Examples include Fougères et al., (2009), where a mixture model is obtained based on a certain number of stable random variables, and Gissibl and Klüppelberg (2018), where a max-linear model is defined on a directed acyclic graph. For the latter, our methods could be useful in the procedure to reconstruct the graph structure of a dataset (Gissibl et al., 2018), since it is based on identifying pairwise tail dependence coefficients that are zero.

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Appendix A. Proofs

Definition (Left/right partial derivatives). Let f be a function whose support includes $\mathcal{X} \subset \mathbb{R}^p$ and let $\mathbf{a} \in \mathcal{X}$. Suppose that $\mathcal{X} - \mathbf{a}$ equals the intersection of a union of orthants and an open cube $C_\varepsilon(\mathbf{0})$ for some $\varepsilon > 0$, i.e., $\mathcal{X} - \mathbf{a}$ is locally equal to a union of orthants. The function f is said to have *left/right (l/r) partial derivatives of order 1* on \mathcal{X} if

1. it has partial derivatives at each interior point of \mathcal{X} ;
2. it has partial derivatives at each boundary point of \mathcal{X} with respect to coordinates that can be perturbed to the left and right;
3. it has left (right) partial derivatives at each boundary point of \mathcal{X} with respect to coordinates that can be perturbed only to the left (right).

The shape of \mathcal{X} is such that for all $\mathbf{x} \in \mathcal{X}$ and for all $i \in \{1, \dots, p\}$, it is possible to perturb x_i to the left, the right, or both and stay within \mathcal{X} . This means that it is always possible to define the left, right, or two-sided partial derivative of f with respect to x_i . We say that f has l/r partial derivatives of order k on \mathcal{X} for $k \geq 2$ if f has l/r partial derivatives of order $k - 1$ on \mathcal{X} and each of the latter has l/r partial derivatives on \mathcal{X} . When we say the f has continuous l/r partial derivatives, continuity is defined in terms of local perturbations within \mathcal{X} only.

Proof of Theorem 3.2. This theorem is a special case of Andrews (1999, Theorem 3b). A closely related work is Andrews (2002), where the focus is on generalized method of moment estimators. This setting is closer to ours but uses a convergence rate of \sqrt{n} , whereas (Andrews, 1999) allows for more generality. The quantities ℓ_T , B_T and R_T in Andrews (1999) correspond to $-(k/2)f_{n,k}$, $\sqrt{k}I_p$ and $-(k/2)R_{n,k}$ respectively in this paper. Theorem 3b in Andrews (1999) holds under the assumptions 1, 2*, 3*, 5*, and 6 of that paper. We show that these are implied by ours:

Assumption 1 The assumptions made in Theorem 3.1 imply assumption 1.

Assumption 2* Assumption GMM2 in Andrews (2002) implies Assumption 2*; this is proven in Andrews (2002, Lemma 3). We show that our assumption imply GMM2: assumption GMM2(a) holds because $D_{n,k}(\boldsymbol{\theta})$ converges in probability to $L(\boldsymbol{\theta}_0) - L(\boldsymbol{\theta})$. Assumption GMM2(b) holds if assumption GMM2*(b) holds, which in turn is implied by the assumptions of Theorem 3.1 and (A1). Assumption GMM2(c) holds since $\mathbf{D}(\boldsymbol{\theta}_0) = \mathbf{0}$. Assumption GMM2(d) holds since $D_{n,k}(\boldsymbol{\theta}) - \mathbf{D}(\boldsymbol{\theta}) - D_{n,k}(\boldsymbol{\theta}_0) = \mathbf{0}$. Finally, Assumption GMM2(e) holds automatically since our weight matrix isn't random.

Assumption 3* Assumption 3* holds because of (A3) and because J is non-random, symmetric and non-singular.

Assumption 5* Assumption 5* holds because of (A2) and because $B_T = \sqrt{k}I_p$ and $\sqrt{k} \rightarrow \infty$ as $n \rightarrow \infty$.

Assumption 6 Assumption (A2) implies assumption 6. \square

Proof of Corollary 3.3. This corollary is a special case of Andrews (1999, Corollary 1b), where no parameter ψ appears. Assumptions 1, 2*, 3*, 5* and 6–8 in that paper are needed: in the proof of Theorem 3.2, we've already shown that our assumptions imply 1, 2*, 3*, 5* and 6; assumptions 7 and 8 hold by our assumption (A4). \square

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